A Study of α-β Filter from Kalman Filter Point

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ABSTRACT

The objective of this paper is to examine whether or not the α-β filter is a version of the Kalman filter. We will closely study the closed-form of the time-variant α-β filter and the stability conditions of the constant α-β filter. Since these problems were studied more than forty years ago, only final conclusions are presented in the recent literature without their derivations or the assumptions on which these derivations depended. Starting with the basic definitions of the α-β filter, its properties are derived in the state space in this paper. The analyses carried out in the time domain and some properties that were not previously reported will be derived. By examining the analysis results and assumptions required for the derivations, similarity or dissimilarity between the two filtering techniques will be concluded.

1.0 Introduction

The α-β filter [16] has been developed and used slightly earlier than the Kalman filtering theory [8]. Compared to the Kalman filter, the structure of the α-β filter is simpler and its computational load is lower. Because of these features, the α-β filter is often used for target tracking by radar systems.

Since both filters has a feedback structure, there has been many attempts to find the relations between the α-β and Kalman filters. Many researchers report the similarity or correspondence of the α-β filter to the Kalman filtering technique [4,7,10,14]. They define the feedback gain matrix in terms of two parameters of α-β and claim that the α-β filter is a modified version of the Kalman filter. It is difficult, however, to confirm their claims, since the properties of various α-β filters are presented without the background derivations or assumptions [4].

We focus in this paper on two classical problems of the α-β filter. Firstly, we derive the closed-form α-β filter starting with the Kalman filter to see whether or not the α-β filter can be generated by modifying the Kalman filter. Secondly, we will examine the steady-state conditions required for the constant α-β filter. The analysis for these problems will clarify the similarities or dissimilarities between the two filtering techniques. The reports that claim the similarity depended on specific assumptions. We itemizes all assumptions that are necessary for the derivation. We will carry out our analysis in the time domain in order to avoid ambiguities/confusions in analyzing stochastic system signals by, for example, *z*-transform [16,14]. A mathematical technique called the Kronecker product [6] will be applied in order to significantly simplify the derivations.

The outline of this paper is as follows. In Section 2.0, a brief introduction to the α-β filter is presented. A closed form α-β filter is derived and its properties are discussed in Section 3.0. The steady-state conditions of the α-β filter are derived in Section 4.0, and the conclusions are presented in Section 5.0.

In this paper, matrices are denoted by capital letters, such as X or A. The transpose of the matrix A is denoted by A^T and the inverse by A^{-1}. All vectors are column vectors denoted in boldface as $\mathbf{x}$.

2.0 Brief introduction to α-β filter

This section is intended for the researchers who are unfamiliar to the α-β filter but have been studying stochastic processes and filtering theory in the state space. The state-space representation presented here will enable them to understand the structure of the α-β filter and its potential similarities to the Kalman filter.

For radar systems, a target is tracked generally in three-dimensional space, e.g., in the *x-y-z* coordinates. The first assumption is to simplify the structure of the α-β filter:

(A1): the target motion in one coordinate is independent from others.

This assumption is significant. It says that whatever happens in the *y* and *z* coordinates, the tracking in the *x*-coordinate is not influenced at all.

The next assumption was made to simplify the analysis. For radars, the sampling intervals are not set equal for all the targets to conserve radar resources. Some targets may require more attention compared to the others. Since we are interested in performance of the basic α-β filter, we stick to the following assumption:

(A2): the sampling interval T_s is constant.

The α-β filter is defined by the equations [2,5,10,14,16]:

$$x_s(k) = x_p(k) + \alpha_k[y_k - x_p(k)] \tag{1}$$

$$v_s(k) = v_s(k-1) + \frac{\beta_k}{T_s}[y_k - x_p(k)] \tag{2}$$

$$x_p(k+1) = x_s(k) + T_s v_s(k), \tag{3}$$

$$v_p(k+1) = v_s(k) \tag{4}$$

where

- $x_s(k)$ = smoothed position
- $v_s(k)$ = smoothed velocity/speed
- $x_p(k)$ = predicted position
- $v_p(k)$ = predicted velocity/speed
- y_k = measurement of position

Using (4), (2) may be replaced with

$$v_s(k) = v_p(k) + \frac{\beta_k}{T_s}[y_k - x_p(k)]. \tag{5}$$

Eliminating $x_s(k)$ from (1), (3),(4), and (5), we get

$$x_p(k+1) = \{x_p(k) + \alpha_k[y_k - x_p(k)]\} + \left\{v_p(k) + \frac{\beta_k}{T_s}[y_k - x_p(k)]\right\}T_s \tag{6}$$

$$v_p(k+1) = v_p(k) + \frac{\beta_k}{T_s}[y_k - x_p(k)]. \tag{7}$$

Simplify the equations, the α-β filter is given in the state-space as:

$$\begin{bmatrix} x_p(k+1) \\ v_p(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_p(k) \\ v_p(k) \end{bmatrix} + \begin{bmatrix} \alpha_k + \beta_k \\ \dfrac{\beta_k}{T_s} \end{bmatrix} \left\{ y(k) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ v_p(k) \end{bmatrix} \right\}. \tag{8}$$

We can also eliminate $x_p(k)$ and $v_p(k)$ from (1), (2), (3), and (4), which results in

$$\begin{bmatrix} x_s(k+1) \\ v_s(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} + \begin{bmatrix} \alpha_k \\ \dfrac{\beta_k}{T_s} \end{bmatrix} \left\{ y(k) - \begin{bmatrix} 1 & T_s \end{bmatrix} \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} \right\}. \tag{9}$$

Many papers do not specify the state-space equations that they assume [5,7,15] or some uses the state-space equations such as [3,4,12]

$$\begin{bmatrix} x_s(k+1) \\ v_s(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} + \begin{bmatrix} \alpha_k \\ \dfrac{\beta_k}{T_s} \end{bmatrix} \left\{ y(k) - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} \right\}. \tag{10}$$

We will use (9) in this paper, since it is derived from the basic filter equations (1)-(5) and closely resembles the Kalman filter. As for (10), we do not think it is valid since it cannot be derived from (1)-(5).

The Kalman filter for the discrete systems are defined, respectively, by

$$\mathbf{x}_{k+1} = \mathrm{A}\mathbf{x}_k + \mathbf{w}_k \tag{11}$$

and

$$y_k = \mathrm{H}\mathbf{x}_k + u_k, \tag{12}$$

where $\mathbf{w}_k$ denotes the white gaussian noise vector with mean zero and covariance matrix R_1 and u_k measurement gaussian noise with mean zeros and variance R_2. The noise sequences $\mathbf{w}_k$ and u_k are assumed independent. The system noise term $\mathbf{w}_k$ is often assumed to represent system model error.

The Kalman filter for (11) and (12) is defined as [1,17]

$$\hat{\mathbf{x}}_{k+1} = \mathrm{A}\hat{\mathbf{x}}_k + \mathrm{K}_k[y_k - \mathrm{H}\hat{\mathbf{x}}_k] \tag{13}$$

where the gain matrix is defined by

$$\mathrm{K}_k = \mathrm{A}\mathrm{P}_k\mathrm{H}^T[\mathrm{H}\mathrm{P}_k\mathrm{H}^T + \mathrm{R}_2]^{-1}, \tag{14}$$

the estimation error covariance matrix is updated according to

$$\mathrm{P}_{k+1} = \mathrm{A}\mathrm{P}_k\mathrm{A}^T + \mathrm{R}_1 - \mathrm{A}\mathrm{P}_k\mathrm{H}^T[\mathrm{H}\mathrm{P}_k\mathrm{H}^T + \mathrm{R}_2]^{-1}\mathrm{H}\mathrm{P}_k\mathrm{A}^T, \tag{15}$$

where we assume here that the initial conditions for (13) and (15) are appropriately set.

Comparing (9) and (13), we can set the following

- the state vector $\mathbf{x}_k = \begin{bmatrix} x(k) & v(k) \end{bmatrix}^T$, where $x(k)$ and $v(k)$ denote the position and velocity/speed of the target at time k
- the estimation output $\hat{\mathbf{x}}_k = \begin{bmatrix} x_s(k) & v_s(k) \end{bmatrix}^T$
- the matrices A and H can be set to

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & T_s \end{bmatrix} \tag{16}$$

- the matrix K_k is set to

$$K_k = \begin{bmatrix} \alpha_k \\ \dfrac{\beta_k}{T_s} \end{bmatrix}. \tag{17}$$

We can confirm that K_k is a 2×1 vector by examining the dimensions of the matrices A and H. Since this is important, we specify it as an assumption:

(A3): the dimensions of A and H are, respectively, 2×2 and 1×2.

We will derive the two classical results concerning the α-β filter in Sections 3.0 and 4.0. Specifically, we will derive the parameters of α_k and β_k in a closed-form depending on the structure of the Kalman filter (13)-(17). Then we will study the conditions required for the α-β parameters in the steady state.

3.0 Closed form α-β filter

There are not many reports that derived the α-β filter directly from the Kalman filter. For example, a closed form solution of the parameters of α and β is considered in [10] where the target motion is assumed to follow a linear track. The solution is derived by minimizing the square error between the estimates and true track.

Since we are examining the connections between the α-β and Kalman filters, we will derive a closed-form solution depending on the Kalman filter equations (13)-(17). A closed-form solution derived form the Kalman filter is found in [4], but only final results are presented without the background derivations. In [12], a closed-form α-β filter is derived from the Kalman filter, assuming (10) as the state space representation.

We adapted, however, the two significant assumptions of [12] in the next section. We will inspect whether or not such assumptions are theoretically and practically acceptable or valid at the end of this section.

3.1 Derivation of close-form α and β parameters

We start introducing an assumption from [12]:

(A4): the system noise is set as $\mathbf{w}_k = 0$ in (11), i.e., $R_1 = 0$.

Set $R_1 = 0$ in (15) and simplify it using the matrix inversion formula (see for examine [17]) as:

$$Q_{k+1} = A^{-T} Q_k A^{-1} + H^T R_2^{-1} H, \tag{18}$$

where the inverse of P_k is set for simplicity to:

$$Q_k = P_k^{-1}. \tag{19}$$

We introduce here the Kronecker product [6] to solve (18) for Q_k. (see Appendix for some properties of the Kronecker product.)

Consider a matrix $X = \begin{bmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_n \end{bmatrix}$, where $\mathbf{x}_l$, $l = 1, \dots, n$ are column vectors. We define the vec operator for X by

$$vec(X) = \begin{bmatrix}\mathbf{x}_1^T \ \dots \ \mathbf{x}_n^T\end{bmatrix}^T. \tag{20}$$

Applying the vec operator to both sides of (18), we get [3]

$$vec(\mathrm{Q}_{k+1}) = (\mathrm{A}^{-T} \otimes \mathrm{A}^{-T})vec(\mathrm{Q}_k) + vec(\mathrm{H}^T\mathrm{R}_2^{-1}\mathrm{H}), \tag{21}$$

where $U \otimes W$ denotes the Kronecker product of matrices of $U = [u_{ij}]$ and W. It is defined by

$$U \otimes W = \begin{bmatrix} u_{11}W & \dots & u_{1n}W \\ \dots & \dots & \dots \\ u_{m1}W & \dots & u_{mn}W \end{bmatrix}, \tag{22}$$

where the dimension of U is assumed as $m \times n$. We can compute:

$$\mathrm{A}^{-T} \otimes \mathrm{A}^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -T_s & 1 & 0 & 0 \\ -T_s & 0 & 1 & 0 \\ T_s^2 & -T_s & -T_s & 1 \end{bmatrix} \text{ and } vec(\mathrm{H}^T\mathrm{R}_2^{-1}\mathrm{H}) = \sigma_v^{-2}\begin{bmatrix}1 \ T_s \ T_s \ T_s^2\end{bmatrix}^T, \tag{23}$$

where the variance of the measurement noise is set to $\mathrm{R}_2 = \sigma_v^2$.

Let us set Q_k to

$$\mathrm{Q}_k = \begin{bmatrix} q_1(k) & q_2(k) \\ q_2(k) & q_3(k) \end{bmatrix}.$$

We can get the equations to update $q_l(k)$, $l = 1, 2, 3$ from (21) and (23) by equating the vector components of the left and right sides of (21) as:

$$q_1(k+1) = q_1(k) + \sigma_v^{-2}$$

$$q_2(k+1) = q_2(k) - T_s q_1(k) + T_s\sigma_v^{-2}$$

$$q_3(k+1) = q_3(k) - 2T_s q_2(k) + T_s^2 q_1(k) + T_s^2\sigma_v^{-2}.$$

Closed form for $q_l(k)$, $l = 1, 2, 3$ can be derived from these equations:

$$q_1(k) = q_1(1) + (k-1)\sigma_v^{-2} \tag{24}$$

$$q_2(k) = q_2(1) - T_s(k-1)q_1(1) - \frac{T_s\sigma_v^{-2}}{2}(k-4)(k-1) \tag{25}$$

$$q_3(k) = q_3(1) - 2T_s(k-1)q_2(1) + T_s^2 q_1(1)(k-1)^2 + \frac{T_s^2\sigma_v^{-2}}{6}(k-1)(2k^2 - 13k + 24). \tag{26}$$

Substituting (24) to (26) into Q_k and inverting it, we can compute the error covariance matrix P_k.

We set the initial conditions for Q_k, following [12], to:

(A5): $q_1(1) = q_2(1) = q_3(1) = 0$.

The reason behind (A5) is the assumption that there is no *a priori* information on the initial state. Thus $P_1 = \infty$. We will discuss this assumption in Section 3.2.

The estimation error covariance matrix P_k can be computed under (A5) as:

$$P_k = 2\sigma_v^2 \begin{bmatrix} \frac{k^2 - 13k + 24}{k^2 - 3k + 2} & \frac{3(k-4)}{k^2 - 3k + 2} \\ \frac{3(k-4)}{k^2 - 3k + 2} & \frac{6}{T_s^2(k^2 - 3k + 2)} \end{bmatrix}. \tag{27}$$

Substitute (27) into the Kalman gain K_k of (14), then the parameters α_k and β_k can be computed as

$$\alpha_k = \frac{4k-6}{k^2+3k-6} \text{ and } \beta_k = \frac{6}{k^2+3k-6} \tag{28}$$

If (8) is used instead of (9), the same closed-form of α_k and β_k as in (28) are derived.

3.2 Examination of the closed form solution

Assuming the structure of the Kalman filter equations (13) - (15), the feedback parameters of α_k and β_k are derived in (28). We would like closely examine the implications of the closed-form solution and assumptions of (A1) - (A5).

(A1) clearly sets a limitation on the applicability of the α-β filter since the three-dimensional coordinates of the target position are not independent even for radar measurements [9]. There is no such constraint for the Kalman filter.

(A3) is crucial for the α-β filter. The observation matrix H must be a row vector. Otherwise the gain matrix K_k cannot be defined in terms of two variables α_k and β_k. Kalman filter does not have such a restriction.

(A4) is also crucial in order to derive closed form solution (28). Recalling that $\mathbf{w}_k$ represents the system model error, this assumption is impractical as mentioned in [12].

(A5) requires a close attention. It implies that $Q_1 = \mathbf{0}$ as the initial condition, which was derived from the assumption that no information on the initial states is available and all components of the estimation error covariance matrix P_1 should be set to ∞. Mathematically, it does not make a sense. Since $P_k Q_k = I$ by definition where I is an identity matrix, $P_1 = \infty$ and $Q_1 = \mathbf{0}$ essentially mean that $\infty \times 0 = I$. It is obvious that such an ∞ matrix cannot be the initial error covariance matrix for (15).

Let us consider the implications of the closed-form solution (28). It is obvious from these equations that both α_k and β_k converge to 0 as $k \to \infty$, indicating that the α-β filter eventually ignores the measurements. The error covariance P_k converges to a 0 matrix as evident in (27), which implies that estimation error will be eventually equal to zero. Unless the closed-form α-β filter converges "quickly" to the true state, the convergence of α_k and β_k to 0 does not make a sense.

Some papers present a closed-form of the α-β filter different from (28). For example, assuming (10) as the state space representation [4], they are given as:

$$\alpha_k = \frac{2(2k+3)}{(k+2)(k+3)} \text{ and } \beta_k = \frac{6}{(k+2)(k+3)}. \tag{29}$$

Both α_k and β_k also converge to 0 as $k \to \infty$ in this case. An interesting thing to notice is either (28) or (29) are not a function of the initial data of P_1 or Q_1. Independence of (28) from the initial data is due to the assumption (A5). As evident in (24)-(26), without this assumption, P_k

would be a function of the initial data. We may conclude that if the closed-form is derived starting with the Kalman filter, it is necessary to make an assumption equivalent to (A5) in order that the closed-form is free of the initial data. Due to such undependable theoretical basis and unsustainable assumptions, the close-form α-β filter is not a trusty filter.

We now turn our attention to the constant α-β filter. Many papers [2,4,9,10,14] focus on its steady state conditions, which is our next topic.

4.0 Steady-state conditions for constant α-β filter

In this section, we assume that both α_k and β_k are constants and real-valued. We derive the necessary and sufficient conditions required for the parameters α and β to satisfy in the steady state.

Equation (9) can be reduced to:

$$\begin{bmatrix} x_s(k+1) \\ v_s(k+1) \end{bmatrix} = \begin{bmatrix} (1-\alpha) & (1-\alpha)T_s \\ -\dfrac{\beta}{T_s} & (1-\beta) \end{bmatrix} \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} + \begin{bmatrix} \alpha \\ \dfrac{\beta}{T} \end{bmatrix} y_{k+1}. \tag{30}$$

Equation (30) was used in [5] to study the steady state conditions of the α-β filter.

Let us set the state vector to:

$$\mathbf{z}(k) = \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix}. \tag{31}$$

Equation (30) can be written as

$$\mathbf{z}(k+1) = \mathrm{F}\mathbf{z}(k) + \mathbf{b}y_{k+1}, \tag{32}$$

where

$$\mathrm{F} = \begin{bmatrix} (1-\alpha) & (1-\alpha)T_s \\ -\dfrac{\beta}{T_s} & (1-\beta) \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} \alpha \\ \dfrac{\beta}{T} \end{bmatrix}. \tag{33}$$

The necessary and sufficient condition for the α-β filter to be bounded-input and bounded-output (BIBO) stable is presented in [10, 16] as

$$0 < \alpha \text{ and } 0 < \beta < 4 - 2\alpha. \tag{34}$$

Since the second inequality implies that $\alpha < 2$, (34) should be interpreted as

$$0 < \alpha < 2 \text{ and } 0 < \beta < 4 - 2\alpha.$$

Let us set $\alpha = 1$ in (30). We get

$$\begin{bmatrix} x_s(k+1) \\ v_s(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\dfrac{\beta}{T_s} & (1-\beta) \end{bmatrix} \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} + \begin{bmatrix} 1 \\ \dfrac{\beta}{T} \end{bmatrix} y_{k+1}.$$

The equation indicates that the position estimate will not fed-back to update it, which implies that the condition $0 < \alpha < 2$ is not sufficient and requires some refinements. It is one of the motivations for the present paper to further investigate into the steady state conditions of the α-β filter.

Let $\mathbf{z}_t$ (2×1) denote the true state corresponding to the estimation vector (31).

We make an assumptions following [5, 16]:

(A6): the true target is updated according to:

$$\mathbf{z}_t(k+1) = \mathrm{F}\mathbf{z}_t(k). \tag{35}$$

This assumption is the same as (A4). Since we need the update equation for the true state, we state it here as an assumption.

(A7): y_k is white gaussian sequence of zero mean and variance σ_y^2.

The frequency response is often employed to examine the filter responses to noise input. In order to compute the frequency response for stationary random signals, the auto- and cross-covariances of the input and output signals are computed. The *z*-transform is then applied to these covariances to compute the frequency spectrums for the output signals and the transfer functions between input and output [13].

In [14,16], (1)-(4) or (30) are regarded as a filter parameterized by α and β with a deterministic input signal, and the *z*-transform was directly applied to compute the transfer function in the *z*-domain. Setting the input y_k to a white noise as in (A7) is to examine the degree to which the so-derived filter in the *z*-domain is affected by noise.

The process of computing the auto- and cross-covariances of the input and output signals and then applying the *z*-transform to them may is often complicated. The reason that the z-transform is applied in [14,16] is to analyze the influence of the feedback loop in (1)-(4) or (30).

In this paper, the state space analysis techniques of linear stochastic systems are applied. It is easier to analyze performance of the stochastic systems in the time-domain, since the analysis of the feedback loops is replaced with the techniques of linear algebra.

4.1 Derivation of output error covariance matrix

The estimation error covariance is derived for the α-β filter in the time domain in this section. The resultant covariance matrix will be used to analyze the stability of the filter.

Let us denote the estimation error by

$$\mathbf{e}(k) = \mathbf{z}(k) - \mathbf{z}_t(k). \tag{36}$$

It follows from (32) and (35) that the error vector $\mathbf{e}(k)$ follows the equation:

$$\mathbf{e}(k+1) = \mathrm{F}\mathbf{e}(k) + \mathbf{b}y_{k+1}. \tag{37}$$

We assume for the initial state:

(A8): the expectation of the initial state is equal to the true initial state:

$$E\{\mathbf{z}(1)\} = \mathbf{z}_t(1). \tag{38}$$

Because of (38), we have $E\{\mathbf{e}(1)\} = 0$. Then it follows from (37) and the assumption (A7) that

$$E\{\mathbf{e}(k)\} = \mathbf{0}. \tag{39}$$

Let W_k denote the covariance matrix of $\mathbf{e}(k)$. It follows from (37), (39), and using (A7) that

$$\mathrm{W}_{k+1} = F\mathrm{W}_k F^T + \mathbf{b}\mathbf{b}^T\sigma_y^2. \tag{40}$$

Equation (40) can be solved for W_k by following the technique of the Kronecker product of Section 3.0. Applying vec operator, we get

$$vec(\mathrm{W}_{k+1}) = (\mathrm{F}\otimes\mathrm{F})vec(\mathrm{W}_k) + vec(\mathbf{b}\mathbf{b}^T)\sigma_y^2, \tag{41}$$

where

$$\mathrm{F} \otimes \mathrm{F} = \begin{bmatrix} (1-\alpha)^2 & (1-\alpha)^2 T_s & (1-\alpha)^2 T_s & (1-\alpha)^2 T_s^2 \\ \frac{-(1-\alpha)\beta}{T_s} & (1-\alpha)(1-\beta) & -\beta(1-\alpha) & (1-\alpha)(1-\beta)T_s \\ \frac{-(1-\alpha)\beta}{T_s} & -\beta(1-\alpha) & (1-\alpha)(1-\beta) & (1-\alpha)(1-\beta)T_s \\ \frac{\beta^2}{T_s^2} & -\frac{\beta(1-\beta)}{T_s} & -\frac{\beta(1-\beta)}{T_s} & (1-\beta)^2 \end{bmatrix} \tag{42}$$

$$vec(\mathbf{b}\mathbf{b}^T) = \begin{bmatrix} \alpha^2 & \frac{\alpha\beta}{T_s} & \frac{\alpha\beta}{T_s} & \frac{\beta^2}{T_s^2} \end{bmatrix}^T . \tag{43}$$

If we assume W_k converges to W as $k \to \infty$. We then have from (41)

$$\mathrm{W} = [I - (\mathrm{F} \otimes \mathrm{F})]^{-1} vec(\mathbf{b}\mathbf{b}^T)\sigma_y^2, \tag{44}$$

where I denotes an 4×4 identity matrix.

We first consider the conditions for the convergence of (41), and then consider the conditions for the converged covariance matrix W to be positive-definite.

4.2 Convergence conditions

In order that $\mathrm{W}_k \to \mathrm{W}$ as $k \to \infty$, the eigenvalues of $\mathrm{F} \otimes \mathrm{F}$ must satisfy

$$|\lambda(\mathrm{F} \otimes \mathrm{F})| < 1 .$$

The eigenvalues λ_i, $i = 1, \ldots, 4$ of $\mathrm{F} \otimes \mathrm{F}$ are computed as

$$\lambda_{1,2} = \left\{ \frac{(\alpha+\beta)^2}{2} - \alpha - 2\beta + 1 \right\} \pm \frac{1}{2}\sqrt{\{(\alpha+\beta)^2 - 4\beta\}(\alpha+\beta-2)^2} \tag{45}$$

$$\lambda_{3,4} = 1 - \alpha . \tag{46}$$

Let us derive the necessary conditions on $|\lambda(\mathrm{F} \otimes \mathrm{F})| < 1$.

It follows from (46) that

$$\alpha > 0 . \tag{47}$$

Two cases can be considered for the eigenvalues λ_1 and λ_2. One is the case where both $\lambda_{1,2}$ are non-real and the other where both are real-valued.

Case1: both $\lambda_{1,2}$ are non-real.

It follows from (45) that the eigenvalues $\lambda_{1,2}$ are non-real only if

$$4\beta > (\alpha+\beta)^2 . \tag{48}$$

Note that in this case λ_1 and λ_2 are conjugate pairs. So, there is no case where one of the eigenvalues is real and the other non-real.

It follows from the requirement $|\lambda(\mathrm{F} \otimes \mathrm{F})| < 1$ that

$$\left\{ \frac{(\alpha+\beta)^2}{2} - \alpha - 2\beta + 1 \right\}^2 + \frac{1}{4}\{4\beta - (\alpha+\beta)^2\}(\alpha+\beta-2)^2 < 1 .$$

The equation can be simplified to:

$$0 < \alpha < 2 . \tag{49}$$

Equation (48) is simplified to

$$(2-\alpha) - 2\sqrt{1-\alpha} < \beta < (2-\alpha) + 2\sqrt{1-\alpha} , \tag{50}$$

where it is assume that

$$\alpha < 1 . \tag{51}$$

Combining (49) and (51), we get

$$0 < \alpha < 1 . \tag{52}$$

Let us now examine the second case where both eigenvalues are real.

Case2: both $\lambda_{1,2}$ are real.

The eigenvalues $\lambda_{1,2}$ are real only if

$$4\beta < (\alpha+\beta)^2 . \tag{53}$$

It follows from the condition $|\lambda(F \otimes F)| < 1$ that

$$\left\{\frac{(\alpha+\beta)^2}{2} - \alpha - 2\beta + 1\right\} - \frac{1}{2}\sqrt{\{(\alpha+\beta)^2 - 4\beta\}(\alpha+\beta-2)^2} > -1 \text{ and} \tag{54}$$

$$\left\{\frac{(\alpha+\beta)^2}{2} - \alpha - 2\beta + 1\right\} + \frac{1}{2}\sqrt{\{(\alpha+\beta)^2 - 4\beta\}(\alpha+\beta-2)^2} < 1 . \tag{55}$$

Equation (53) simplified to

$$\beta < (2-\alpha) - 2\sqrt{1-\alpha} \text{ or } \beta > (2-\alpha) + 2\sqrt{1-\alpha} , \tag{56}$$

where it is assumed that

$$\alpha < 1 .$$

Equation (54) is reduced to the following two conditions:

$$\frac{(\alpha+\beta)^2}{2} - \alpha - 2\beta + 2 > 0 \text{, and}$$

$$\left\{\frac{(\alpha+\beta)^2}{2} - \alpha - 2\beta + 2\right\}^2 - \frac{1}{4}\{(\alpha+\beta)^2 - 4\beta\}(\alpha+\beta-2)^2 > 0 .$$

Simplifying these inequalities, we get

$$\{\beta - (2-\alpha)\}^2 + 2\alpha > 0 \text{ and}$$

$$\{\beta - (2-\alpha)\}^2 + \alpha^2 > 0 .$$

Since $\alpha > 0$, these two inequalities hold and can be ignored.

Similarly we have from (55) that

$$-\frac{(\alpha+\beta)^2}{2} + \alpha + 2\beta > 0 \text{, and}$$

$$\{(\alpha+\beta)^2-4\beta\}(\alpha+\beta-2)^2 < 4\left\{-\frac{(\alpha+\beta)^2}{2}+\alpha+2\beta\right\}^2 .$$

After simplification of these two inequalities, we get

$$(2-\alpha)-\sqrt{2(2-\alpha)} < \beta < (2-\alpha)+\sqrt{2(2-\alpha)}\text{, and} \tag{57}$$

$$\beta(-2\alpha-\beta+4) > 0, \tag{58}$$

where it is assumed that

$$\alpha < 2. \tag{59}$$

Since $2\sqrt{1-\alpha} < \sqrt{2(2-\alpha)}$ holds if $\alpha > 0$, (56) and (57) are reduced, respectively, to

$$(2-\alpha)-\sqrt{2(2-\alpha)} < \beta < (2-\alpha)-2\sqrt{1-\alpha} \text{ and} \tag{60}$$

$$(2-\alpha)+2\sqrt{1-\alpha} < \beta < (2-\alpha)+\sqrt{2(2-\alpha)}. \tag{61}$$

Let us summarize the conditions derived for $|\lambda(\mathrm{F}\otimes\mathrm{F})| < 1$.

Case1: both $\lambda_{1,2}$ are non-real.

$$(2-\alpha)-2\sqrt{1-\alpha} < \beta < (2-\alpha)+2\sqrt{1-\alpha} \text{ and} \tag{62}$$

$$0 < \alpha < 1 \tag{63}$$

Case2: both $\lambda_{1,2}$ are real.

$$0 < \alpha < 1 \text{ and} \tag{64}$$

$$\beta(-2\alpha-\beta+4) > 0 \text{ and} \tag{65}$$

$$\begin{array}{c}(2-\alpha)-\sqrt{2(2-\alpha)} < \beta < (2-\alpha)-2\sqrt{1-\alpha} \\ \text{or} \\ (2-\alpha)+2\sqrt{1-\alpha} < \beta < (2-\alpha)+\sqrt{2(2-\alpha)}\end{array}. \tag{66}$$

Now let us consider the requirements that W is positive-definite.

4.3 Positive-definite conditions for W

Let us denote the (i,j)-th component of W by W_{ij}. Substituting (42) and (43) into (44), we get

$$\mathrm{W}_{11} = \frac{2\alpha^2-3\alpha\beta+2\beta}{\alpha(4-2\alpha-\beta)}\sigma_y^2 \tag{67}$$

$$\mathrm{W}_{12} = \mathrm{W}_{21} = \frac{\beta(2\alpha-\beta)}{\alpha(4-2\alpha-\beta)T_s}\sigma_y^2 \tag{68}$$

$$\mathrm{W}_{22} = \frac{2\beta^2}{\alpha(4-2\alpha-\beta)T_s^2}\sigma_y^2. \tag{69}$$

Let us derive obvious constraints from (67) and (69).

Since W_{22} should be positive, we have

$$\alpha(4-2\alpha-\beta) > 0 \tag{70}$$

Since W_{11} should be positive and (70) holds, we have

$$(2\alpha^2-3\alpha\beta+2\beta) > 0. \tag{71}$$

Since $\alpha > 0$, we have from (70) that

$$\beta < 4 - 2\alpha\,. \tag{72}$$

The constraint (71) can be reduced to

$$\begin{aligned} \beta &< \frac{2\alpha^2}{3\alpha - 2} \text{ if } \alpha > \frac{2}{3} \\ \beta &> \frac{2\alpha^2}{3\alpha - 2} \text{ if } \alpha < \frac{2}{3} \end{aligned}\,. \tag{73}$$

The matrix W is positive definite if and only if

$$\mathbf{v}^T \mathrm{W} \mathbf{v} > 0$$

holds for an arbitrary 2×1 vector $\mathbf{v}$.

Let us set $\mathbf{v} = [v_1, v_2]^T$. Substituting (67)-(69) into W, we get after simplification

$$\mathbf{v}^T \mathrm{W} \mathbf{v} = \left[\frac{2\alpha^2 - 3\alpha\beta + 2\beta}{\alpha(4 - 2\alpha - \beta)}\left\{v_1 + \frac{\beta(2\alpha - \beta)}{T_s(2\alpha^2 - 3\alpha\beta + 2\beta)}\right\}^2 + \frac{\beta^3}{T_s^2\alpha(2\alpha^2 - 3\alpha\beta + 2\beta)}v_2^2\right]\sigma_y^2\,. \tag{74}$$

Since $\alpha > 0$ and (70) - (72) hold, it is easy to see from (74) that

$$\beta > 0\,. \tag{75}$$

Let us summarize the conditions for W to be a positive-definite matrix:

$$\alpha > 0 \text{ and} \tag{76}$$

$$0 < \beta < 4 - 2\alpha \text{ and} \tag{77}$$

$$\begin{aligned} \beta &< \frac{2\alpha^2}{3\alpha - 2} \text{ if } \alpha > \frac{2}{3} \\ \beta &> \frac{2\alpha^2}{3\alpha - 2} \text{ if } \alpha < \frac{2}{3} \end{aligned}\,. \tag{78}$$

We next combine the results of Sections 4.2 and 4.3 to derive the necessary and sufficient conditions for stable and positive-definite W.

4.3.1 Conditions on convergent error matrix W

It can be shown that the following inequalities hold under $0 < \alpha < 1$:

$$(2 - \alpha) - \sqrt{2(2 - \alpha)} > 0 \tag{79}$$

and

$$(2 - \alpha) + 2\sqrt{1 - \alpha} < 4 - 2\alpha < (2 - \alpha) + \sqrt{2(2 - \alpha)}\,. \tag{80}$$

and

$$\begin{aligned} 4 - 2\alpha &< \frac{2\alpha^2}{3\alpha - 2} \text{ if } \frac{2}{3} < \alpha < 1 \\ \frac{2\alpha^2}{3\alpha - 2} &< 0 \quad \text{ if } 0 < \alpha < \frac{2}{3} \end{aligned}\,. \tag{81}$$

Using (79)-(81), we can simplify the conditions for positive definiteness and convergence for the matrix W.

Case 1:

$$(2-\alpha)-2\sqrt{1-\alpha}<\beta<(2-\alpha)+2\sqrt{1-\alpha} \text{ and}$$
$$0<\alpha<1$$

Case2:

$$\left.\begin{array}{c}(2-\alpha)-\sqrt{2(2-\alpha)}<\beta<(2-\alpha)-2\sqrt{1-\alpha}\\ \text{or}\\ (2-\alpha)+2\sqrt{1-\alpha}<\beta<4-2\alpha\end{array}\right. \quad \text{and}$$
$$0<\alpha<1$$

Combining Cases 1 and 2, we get

$$(2-\alpha)-\sqrt{2(2-\alpha)}<\beta<4-2\alpha \quad \text{and } 0<\alpha<1\,. \tag{82}$$

The BIBO conditions (76) and (77) presented in [10] are refined to (82) by considering the convergence requirement for (40) and positive definiteness of the converged error matrix W.

4.3.2 Examination of the steady-state α-β filter

In order that the steady error covariance converges $\mathrm{W}_k \to \mathrm{W}$ as $k \to \infty$, we have to considered:

- Does (40) converges as $k \to \infty$?
- Is the convergent error covariance W positive-definite?

These considerations resulted in the conditions (82), which is a refinement of the BIBO stability condition (34).

Note that (82) is derived under the assumptions (A5) and (A6). According to (A6), the measurements y_k is set to a white Gaussian noise sequence. This assumption is equivalent to setting the measurement matrix $\mathrm{H} = \mathbf{0}$ in the definition of the discrete system (12) and (13). If $\mathrm{H} = \mathbf{0}$ is substituted in (14) and (15), $\mathrm{K} = \mathbf{0}$ results and the Kalman filter loses the feedback loop. Such a filter cannot be called a version of the Kalman filter.

The system noise $\mathbf{w}_k$ in (12) is also ignored under (A5). Note that one of the features of the Kalman filter is that it can estimate/track the states of the stochastic systems.

5.0 Conclusions

We derived the closed-form α-β filter starting with the Kalman filter equations. We specified the assumptions that are used to drive α_k and β_k as a function of only sampling time. As shown in Section 3.0, the derivation depended on these unsustainable assumptions. *We conclude that the closed-form α-β filter cannot be a version of the Kalman filter.*

The stability conditions of the α-β filter was derived by setting the measurement term y_k equal to a white gaussian noise, following [16]. As pointed out in Section 4.3.2, such an assumption disregards the fundamental structure of the Kalman filter.

It is concluded in this paper that the steady state α-β filter should be considered as a completely different filtering technique from the Kalman filter if the α-β values are selected to satisfy (82).

It should be pointed out that there are techniques to compute the steady state feed-back gain matrix K of the Kalman filter in a closed-from [11] without the assumptions (A1), (A3), (A4) (A5), and (A7). Computational complexity is significantly high for the steady gain matrix of the

Kalman filter compared to the steady state α-β filter. Since the computation can be carried out off-line, the computational cost is not a practical issue.

It is important from the engineering point of view, however, to determine which estimation technique should be selected, the Kalman filter or α-β filter or other filters. It is beyond the scope of the present paper to decide which technique should be used. Since computational cost is presently rapidly reducing, estimation accuracy would be the primary concern. Estimation accuracy will be determined by the accuracy how closely the system model describes the ground truth.

Finally, we would like to point out that Kronecker product significantly simplify the analysis of feedback loop. Complicated derivation of the *z*-transforms of the auto- and cross-covariances of the input and output can be completely eliminated and enables to directly examine the influence of the random process on the estimation algorithms.

6.0 References

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Appendix

Since Kronecker product may be unfamiliar, properties that are used in this paper are presented here as lemmas without proof. It is assumed that the dimensions of all matrices A, B, C and X in this appendix are appropriately defined. The proofs of all of these lemmas are found in many text books (e.g., [6].)

Lemma 1. $vec(A+B) = vecA + vecB$

Lemma 2. $(\alpha A) \otimes B = A \otimes (\alpha B)$, where α is a scalor.

Lemma 3. The matrix equation $AXB = C$ is equivalent to $(B^T \otimes A)vec(X) = vec(C)$.