Useful Formulas of the Kalman Filter with Uncorrelated Noise Elements in Measurement Equations

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ABSTRACT

We show useful formulas of the Kalman filter with uncorrelated noise elements in measurement equations. These formulas show that the Kalman filtering estimtes at each time can be computed by each independent (i.e. uncorrelated) measurement component data recursively. Namely each measurement with each uncorrelated noise component is used separately by one by one for computing Kalman filtering. Furthermore, we derive the measurement update formula of summarize each measurement update estimate and error covariance by using each measurement with each uncorrelated noise component.

1 Kalman filter - introduction

The Kalman filter [1]-[5], is the minimum mean square error (minimum error covariance) filter based on the measurement $Y^t := \{y_0, y_1, \cdots, y_t\}$ to estimate the state x_t as follows:

$$\begin{aligned} x_{t+1} &= F_t x_t + w_t, \quad t = 0, 1, \cdots \quad \text{(State equation)}, \quad x_t : (n \times 1), && (1) \\ & \mathrm{E}[\mathrm{x}_0] = \hat{\mathrm{x}}_0 = \hat{\mathrm{x}}_{0|-1}, \quad \mathrm{Cov}[\mathrm{x}_0] = \Sigma_0 = \Sigma_{0|-1}, && (2) \\ y_t &= H_t x_t + v_t, \quad t = 0, 1, \cdots \quad \text{(Measurement equation)}, \quad y_t : (m \times 1), && (3) \end{aligned}$$

where w_t and v_t are mutually independent Gaussian white noises such as

$$\begin{aligned} \mathrm{E}[w_t] &= 0, \quad \mathrm{E}[w_t w_\tau] = Q_t \delta_{t-\tau}, && (4) \\ \mathrm{E}[v_t] &= 0, \quad \mathrm{E}[v_t v_\tau] = R_t \delta_{t-\tau}. \quad (\text{where } \delta_t \text{ is the Kronecker's } \delta - \text{function}). && (5) \end{aligned}$$

It is well known the minimum error variance estimate of x_t based on the measurement Y^t is:

$$\arg\min_{\hat{x}} \mathrm{E}\Big[\ ||x_t - \hat{x}||^2\ \Big| Y^t\Big] \quad = \quad \mathrm{E}\Big[x_t\ \Big|\ Y^t\Big] := \hat{x}_{t|t}$$

Namely the minimum variance estimate is the conditional expectation of x_t by given measurements $Y^t := \{y_0, y_1, \cdots, y_t\}$. The conditional expectation of x_t; $\hat{x}_{t|t}$, and the conditional error covariance; $\Sigma_{t|t}$, are given [4], [5] by

$$\begin{aligned} \hat{x}_{t|t} &:= (\Sigma_{t|t-1}^{-1} + H_t^{\mathrm{T}} R_t^{-1} H_t)^{-1} (\Sigma_{t|t-1}^{-1} \hat{x}_{t|t-1} + H_t^{\mathrm{T}} R_t^{-1} y_t) && (6) \\ \Sigma_{t|t} &:= (\Sigma_{t|t-1}^{-1} + H_t^{\mathrm{T}} R^{-1} H_t)^{-1}. && (7) \end{aligned}$$

Then by applying the matrix inversion lemmas, we have well known recursive equations:

$$\begin{aligned}
\hat{x}_{t|t} &= \hat{x}_{t|t-1} + K_t(y_t - H_t\hat{x}_{t|t-1}), && (8)\\
&\quad K_t := \Sigma_{t|t-1}H_t^{\mathrm{T}}(H_t\Sigma_{t|t-1}H_t^{\mathrm{T}} + R_t)^{-1}, \quad (\text{: Kalman Gain}) && (9)\\
\hat{x}_{t+1|t} &= F_t\hat{x}_{t|t}, && (10)\\
\Sigma_{t|t} &= \Sigma_{t|t-1} - K_tH_t\Sigma_{t|t-1}, && (11)\\
\Sigma_{t+1|t} &= F_t\Sigma_{t|t}F_t^{\mathrm{T}} + Q_t. && (12)
\end{aligned}$$

2 Uncorrelated noise elements in measurement eqs.

Here we consider the special case of the measurement equations as follows:

$$y_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} H_{1,t} \\ H_{2,t} \end{bmatrix} x_t + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix} := H_tx_t + v_t$$

Namely, we assume that $v_{1,t}$ and $v_{2,t}$ are mutually uncorrelated (independent) white Gaussian noises with zero means. Namely

$$\mathrm{E}[v_{1,t}] = 0, \quad \mathrm{E}[v_{2,t}] = 0, \tag{13}$$

$$\mathrm{E}\Big\{ \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix} \begin{bmatrix} v_{1,\tau}^{\mathrm{T}} & v_{2,\tau}^{\mathrm{T}} \end{bmatrix} \Big\} = \begin{bmatrix} R_{1,t}\delta_{t-\tau} & O \\ O & R_{2,t}\delta_{t-\tau} \end{bmatrix} \tag{14}$$

where δ_t is Kronecker's delta function. Then we will prove that the following Theorem.

Theorem 1 *Let* $\hat{x}_{t|t} := \mathrm{E}[x_t|Y^t]$, $\hat{x}_{t|t-1} := \mathrm{E}[x_t|Y^{t-1}]$, $\hat{x}_{t|t-1,1} := \mathrm{E}[x_t|Y^{t-1}, y_{1,t}]$ *and* $\Sigma_{t|t} := \mathrm{Cov}[x_t|Y^t]$, $\Sigma_{t|t-1} := \mathrm{Cov}[x_t|Y^{t-1}]$, $\Sigma_{t|t-1,1} := \mathrm{Cov}[x_t|Y^{t-1}, y_{1,t}]$,, *then*

$$\begin{aligned}
\hat{x}_{t|t} &= (\Sigma_{t|t-1,1}^{-1} + H_{2,t}^{\mathrm{T}}R_{2,t}^{-1}H_{2,t})^{-1}(\Sigma_{t|t-1,1}^{-1}\ \hat{x}_{t|t-1,1} + H_{2,t}^{\mathrm{T}}R_{2,t}^{-1}y_{2,t})\\
&= \hat{x}_{t|t-1,1} + \Sigma_{t|t-1,1}H_{2,t}^{\mathrm{T}}(H_{2,t}\Sigma_{t|t-1,1}H_{2,t}^{\mathrm{T}} + R_{2,t})^{-1}(y_{2,t} - H_{2,t}\hat{x}_{t|t-1,1}), && (15)\\
\Sigma_{t|t} &= [\Sigma_{t|t-1,1}^{-1} + H_{2,t}^{\mathrm{T}}R_{2,t}^{-1}H_{2,t}]^{-1}\\
&= \Sigma_{t|t-1,1} - \Sigma_{t|t-1,1}^{-1}H_{2,t}(H_{2,t}\Sigma_{t|t-1,1}H_{2,t}^{\mathrm{T}} + R_{2,t}^{-1})H_{2,t} && (16)
\end{aligned}$$

where

$$\begin{aligned}
\hat{x}_{t|t-1,1} &= \hat{x}_{t|t-1} + \Sigma_{t|t-1}H_{1,t}^{\mathrm{T}}(H_{1,t}\Sigma_{t|t-1}H_{1,t}^{\mathrm{T}} + R_{1,t})^{-1}(y_{1,t} - H_{1,t}\hat{x}_{t|t-1}) && (17)\\
\Sigma_{t|t-1,1} &= \Sigma_{t|t-1} - \Sigma_{t|t-1}^{-1}H_{1,t}(H_{1,t}\Sigma_{t|t-1,1}H_{1,t}^{\mathrm{T}} + R_{1,t}^{-1})H_{1,t}. && (18)
\end{aligned}$$

Proof 1 *Let us evaluate the following conditional expectation:*

$$\mathrm{E}[x_t|Y^{t-1}, y_{1,t}] := \hat{x}_{t|t-1,1}.$$

From (6) and (7), we have

$$\hat{x}_{t|t-1,1} = (\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}}R_1^{-1}H_{1,t})^{-1}(\Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}}R_{1,t}^{-1}y_{1,t}) \tag{19}$$

and

$$\Sigma_{t|t-1,1} = (\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_1^{-1} H_{1,t})^{-1} = \mathrm{Cov}[x_t | Y^{t-1}, y_{1,t}]. \tag{20}$$

Furthemore, we can evaluate the following conditional expectation:

$$\hat{x}_{t|t} := \mathrm{E}[x_t | Y^{t-1}, y_{1,t}, y_{2,t}] = \mathrm{E}[x_t | Y^t],$$

as

$$\hat{x}_{t|t} = (\Sigma_{t|t-1,1}^{-1} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} H_{2,t})^{-1} (\Sigma_{t|t-1,1}^{-1}\ \hat{x}_{t|t-1,1} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}) \tag{21}$$

$$\Sigma_{t|t} = [\Sigma_{t|t-1,1}^{-1} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} H_{2,t}]^{-1} \tag{22}$$

Finally, from (21) with (19), (20), (22), we have

$$\begin{aligned}
\hat{x}_{t|t} &= (\Sigma_{t|t-1,1}^{-1} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} H_{2,t})^{-1} \Big[\Sigma_{t|t-1,1}^{-1}\ \hat{x}_{t|t-1,1} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}\Big] \\
&= (\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} H_{1,t} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} H_{2,t})^{-1} \\
&\quad \times \Big[(\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_1^{-1} H_{1,t})(\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} H_{1,t})^{-1} \\
&\quad \times (\Sigma_{t|t-1} \hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} y_{1,t}) + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}\Big] \\
&= (\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} H_{1,t} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} H_{2,t})^{-1} \\
&\quad \times \Big[\Sigma_{t|t-1} \hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} y_{1,t} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}\Big] \\
&= \Big(\Sigma_{t|t-1}^{-1} + \begin{bmatrix} H_{1,t} \\ H_{2,t} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_{1,t} & O \\ O & R_{2,t} \end{bmatrix}^{-1} \begin{bmatrix} H_{1,t} \\ H_{2,t} \end{bmatrix} \Big)^{-1} \\
&\quad \times \Big(\Sigma_{t|t-1}^{-1} \hat{x}_{t|t-1} + \begin{bmatrix} H_{1,t} \\ H_{2,t} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_{1,t} & O \\ O & R_{2,t} \end{bmatrix}^{-1} \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} \Big)
\end{aligned} \tag{23}$$

The above equation coincides to Eq. (6)

Corollary 1 *Let* $\hat{x}_{t|t,2} := \mathrm{E}[x | Y^{t-1}, y_{2,t}]$ *and* $\Sigma_{t|t-1,2} := \mathrm{Cov}[x_t | Y^{t-1}, y_{2,t}]$, *then*

$$\begin{aligned}
\hat{x}_{t|t} &= \Big[\Sigma_{t|t-1,1}^{-1} + \Sigma_{t|t-1,2}^{-1} - \Sigma_{t|t-1}^{-1}\Big]^{-1} \\
&\quad \times \Big[\Sigma_{t|t-1,1}^{-1} \hat{x}_{t|t-1,1} + \Sigma_{t|t-1,2}^{-1} \hat{x}_{t|t-1,2} - \Sigma_{t|t-1}^{-1} \hat{x}_{t|t-1}\Big]
\end{aligned} \tag{24}$$

and

$$\Sigma_{t|t} = \Big[\Sigma_{t|t-1,1}^{-1} + \Sigma_{t|t-1,2}^{-1} - \Sigma_{t|t-1}^{-1}\Big]^{-1} \tag{25}$$

Proof 2 *From (20), we have*

$$\begin{aligned}
&\Big[\Sigma_{t|t-1,1}^{-1} + \Sigma_{t|t-1,2}^{-1} - \Sigma_{t|t-1}^{-1}\Big]^{-1} \\
&= \Big[\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} H_{1,t} + \Sigma_{t|t-1}^{-1} + H_{2,t}^{\mathrm{T}} R_{1,t}^{-1} H_{2,t} - \Sigma_{t|t-1}^{-1}\Big]^{-1} \\
&= \Big[\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} H_{1,t} + H_{2,t}^{\mathrm{T}} R_{1,t}^{-1} H_{2,t}\Big]^{-1} \\
&= \Sigma_{t|t}
\end{aligned} \tag{26}$$

and also applying the relation in (19), we have

$$\hat{x}_{t|t-1,1} = (\Sigma_{t|t-1,1})(\Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} y_{1,t}) \tag{27}$$
$$\hat{x}_{t|t-1,2} = (\Sigma_{t|t-1,2})(\Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}), \tag{28}$$

therefore, we have the following relations:

$$\begin{aligned} & \left[\Sigma_{t|t-1,1}^{-1}\hat{x}_{t|t-1,1} + \Sigma_{t|t-1,2}^{-1}\hat{x}_{t|t-1,2} - \Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1}\right] \\ = & \left[\Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} y_{1,t} + \Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{2,t}^{\mathrm{T}} R_{1,t}^{-1} y_{2,t} - \Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1}\right] \\ = & \left[\Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} y_{1,t} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}.\right] \end{aligned} \tag{29}$$

From (28) and (29), the righthand side in (24) becomes

$$\begin{aligned} & \left[\Sigma_{t|t-1}^{-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} H_{1,t} + H_{2,t}^{\mathrm{T}} R_{1,t}^{-1} H_{2,t}\right]^{-1} \\ & \times\left[\Sigma_{t|t-1}^{-1}\hat{x}_{t|t-1} + H_{1,t}^{\mathrm{T}} R_{1,t}^{-1} y_{1,t} + H_{2,t}^{\mathrm{T}} R_{2,t}^{-1} y_{2,t}\right. \\ = & \ \hat{x}_{t|t}. \end{aligned} \tag{30}$$

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