Novel Extended Kalman Filters with Fixed-Point Smoothers

by

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ABSTRACT

We show the novel extended Kalman filter with using the fixed-point smoother algorityym. The extended Kalman fitering algorithms are most popular for GNSS positioning. So-called GR models and the corresponding positioning algorithms are closed connected for the extended Kalman filters. In this paper, we show the novel extended Kalman Kalman fiters with applying the linear fixed-point smoother algorithms.

1 The Extended Kalman Filter (EKF) - Introduction

The extended Kalman filter [1]-[3] are derived by applying the first order Taylor series expansion for the following nonlinear state equation $x_k(:\ n \times 1)$, and the measurement $y_k(:\ m \times 1)$ equation as follows:

$$\begin{aligned} x_{k+1} &= f_k(x_k) + g_k(x_k)w_k \quad k = 0, 1, \cdots \text{(Nonlinear state equation)} && (1) \\ & \quad \text{InitialConditions : } \mathrm{E}[x_0] = \hat{x}_0 = \hat{x}_{0|-1}, \quad \mathrm{Cov[x_0]} = \Sigma_0 = \Sigma_{0|-1}, && (2) \\ y_k &= h_k(x_k) + v_k \quad k = 0, 1, \cdots \text{(Nonlinear measurement equation)}, && (3) \end{aligned}$$

where w_k, and v_k are mutually independent Gaussian white noise such as

$$\begin{aligned} \mathrm{E}[w_k] &= 0, \quad \mathrm{E}[w_k w_\tau] = Q_k \delta_{t-\tau}, && (4) \\ \mathrm{E}[v_k] &= 0, \quad \mathrm{E}[v_k v_\tau] = R_k \delta_{t-\tau}. \quad \text{(where } \delta_k \text{ is the Kronecker's } \delta - \text{function)} && (5) \end{aligned}$$

It is well known the minimum error variance estimate of x_k based on the measurement Y^k is:

$$\arg\min_{\hat{x}_k} \mathrm{E}\Big[\ ||x_k - \hat{x}_k||^2\ \Big| Y^k\Big] \ = \ \mathrm{E}\Big[x_k\ \Big|\ Y^k\Big] := \hat{x}_{k|k} \qquad (6)$$

Namely the minimum variance estimate is the conditional expectation of x_k based on the given mesurements Y^k. To obtain the conditional expectation $\mathrm{E}\Big[x_k\ \Big|\ Y^k\Big]$ by the Kalman filtering formula.

First of all, we define the matrices as the partial derivatives:

$$F_k^{\mathrm{T}} := \frac{\partial f_k(x)}{\partial x}\Big|_{x=\hat{x}_{k|k}}, \quad H_k^{\mathrm{T}} := \frac{\partial h_k(x)}{\partial x}\Big|_{x=\hat{x}_{k|k-1}}, \quad (\cdot)^{\mathrm{T}} : \text{ Transpose of } (\cdot), \qquad (7)$$

and the matrix:

$$G_k := g_k(\hat{x}_{k|k}) \qquad (8)$$

Then the following first-order Taylor's series approximations are applied for the nonlinear functions as follows:

$$\begin{aligned}
f_k(x_k) &= f_k(\hat{x}_{k|k}) + F_k[x_k - \hat{x}_{k|k}] + \cdots \\
&\approx F_k x_k + f_k(\hat{x}_{k|k}) - F_k \hat{x}_{k|k} \\
&:= F_k x_k + s_k \qquad (9) \\
g_k(x_k) &= [g_k(\hat{x}_{k|k}) + \cdots \\
&\approx G_k \qquad (10) \\
h_k(x_k) &= h_k(\hat{x}_{k|k-1}) + H_k[x_k - \hat{x}_{k|k-1}] + \cdots \\
&\approx H_k x_k + h_k(\hat{x}_{k|k-1}) - H_k \hat{x}_{k|k-1} \\
&:= H_k x_k + r_k \qquad (11)
\end{aligned}$$

Therefore (1) and (3) are approximately described by

$$\begin{aligned}
x_{t+1} &\approx F_k x_k + s_k + G_k w_k \qquad (12) \\
&\mathrm{E}[\mathrm{x}_0] = \hat{\mathrm{x}}_0 = \hat{\mathrm{x}}_{0|-1}, \quad \mathrm{Cov}[\mathrm{x}_0] = \Sigma_0 = \Sigma_{0|-1}, \qquad (13) \\
y_k &\approx H_k x_k + r_k + v_k \qquad (14)
\end{aligned}$$

where

$$\begin{aligned}
s_k &= f_k(\hat{x}_{k|k}) - F_k \hat{x}_{k|k} \qquad (15) \\
r_k &= h_k(\hat{x}_{k|k-1}) - H_k \hat{x}_{k|k-1} \qquad (16) \\
& \qquad (17)
\end{aligned}$$

Therefore the corresponding Kalman filters [4], [1], [5] are given by

$$\begin{aligned}
\hat{x}_{k+1|k} &= F_k \hat{x}_{k|k} + s_k \qquad (18) \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - H_k \hat{x}_{k|k-1} - r_k), \quad k = 0,\ 1,\ \ldots \qquad (19) \\
&K_k = F_k \Sigma_{k|k-1} H_k^T \left[H_k \Sigma_{k|k-1} H_k^T + R_k \right]^{-1}, \quad ((\text{Kalman Gain})) \qquad (20) \\
\Sigma_{k+1|k} &= F_k \Sigma_{k|k} F_k^T + G_k Q_k G_k^T \qquad (21) \\
\Sigma_{k|k} &= \Sigma_{k|k-1} - K_k H_k \Sigma_{k|k-1}, \quad k = 0,\ 1,\ \ldots \qquad (22) \\
&\mathrm{IC}:\ \hat{x}_{0|-1} = \hat{x}_0, \quad \Sigma_{0|-1} = \Sigma_0 \qquad (23)
\end{aligned}$$

2 EKF with the Fixed-Point Smoother (FPS)

Obviously, the accuracy of filtering estimates by applying the extended Kalman filter, is depended on the accuracy of the initial estimates of $\hat{x}_0 := \hat{x}_{0|-1}$ and $\Sigma_{0|-1} = \Sigma_0$, because the Taylor expansion around; $\hat{x}_{0|0}$ for $f_k(x_0)$, and $\hat{x}_{0|-1}$ for $h_k(x_0)$ are applied. Thus, we try to apply the fixed-point smoother(FPS) for getting more accurated initial estimates $\hat{x}_{0|L-1}$, (L is a positive integer), instead of $\hat{x}_{0|-1}$. Namely, we combine the extended Kalman filter for real-time (on line) computation for filtering, and each L-period we applying the fixed-point smoother for providing more accurated initial estimates. The following fixed-point smoother [1]-[3] are applied for $E(x_j|Y^k),\ j \le k$.

Recursive algorithms for fixed-point smoothers are given by the following steps ([2]): (PS1), (PS2) and (PS3).

1. (PS1) Smoothing Estimates:

$$\begin{aligned}\hat{x}_{j|k} &= \hat{x}_{j|k-1} + K_{P,k}(j)[y_k - H_k\hat{x}_{k|k-1} - r_k], \quad k = j, j+1, \cdots &(24)\\ &= \hat{x}_{j|k-1} + K_{P,k}(j)\nu_k, \quad \nu_k := y_k - H_k\hat{x}_{k|k-1} - r_k &(25)\end{aligned}$$

2. (PS2) Smoothing Gain:

$$K_{P,k}(j) = \Omega_{k|k-1}H_k^T[H_k\Sigma_{k|k-1}H_k^T + R_k]^{-1} \; k = j, j+1, \cdots \tag{26}$$

3. (PS3) Covariance matrix of Smoothing error:

$$\Omega_{k+1|k} = \Omega_{k|k-1}[I - K_kH_k]^TF_k^T, \quad k = j, j+1, \cdots \tag{27}$$

$$\Sigma_{j|k} = \Sigma_{j|k-1} - \Omega_{k|k-1}H_k^T[H_k\Sigma_{k|k-1}H_k^T + R_k]^{-1}H_k\Omega_{k|k-1}^T \tag{28}$$

where the boundary condition is given by

$$\Omega_{j|j-1} = \Sigma_{j|j-1}, \tag{29}$$

and $\hat{x}_{k|k-1}, \Sigma_{k|k-1}, K_k$ are given by the Kalman filters in (18), (21) and (20)

The following novel EKF with the fixed-point smoother are proposed:
Let $q = 0$ and $k = qL, qL+1, \cdots, (q+1)L-1$, ($L$: positive integer),

1. (EKF and FPS) Let $k = qL, \cdots, (q+1)L-1$
and obtain $\hat{x}_{k|k}$ by EKF in (18)-(22)
and $\hat{x}_{qL|(q+1)L-1}$ by FPS in (24)-(28).
Then provide $\hat{x}_{k|k}$ for $k = qL, \cdots, (q+1)L-1$ to users.

2. (EKF and FPS with renewing IC) Obtain $\hat{x}_{k|k}$ by EKF in (18)-(22)
for $k = qL, \cdots, (q+2)L-1$,
with renewing IC: $\hat{x}_{qL|qL-1} := \hat{x}_{qL|(q+1)L-1}$, $\Sigma_{qL|qL-1}; = \Sigma_{qL|(q+1)L-1}$.
Simultaneously, compute $\hat{x}_{(q+1)L|(q+2)L-1}$ by FPS in (24)-(28).
Provide $\hat{x}_{k|k}$ for $k = (q+1)L, \cdots, (q+2)L-1$ to users.

3. (Let $q = q+1$, Go to 1)

References

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