Optimal Design of α-β-(γ) Filters

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Abstract

Optimal sets of the smoothing parameter (α, β and γ) are derived for a sampled data target tracker. A constrained parameter optimization problem is formulated for specific trajectories of the target, which includes the noise in the measurement, the steady state error and the transient response of the filter. This work considers two classes of target trajectories, circular and straight line maneuvers. Closed form expressions for the steady state error and the sum of the square of the errors capturing the transient behavior are derived for a straight line trajectory where the target moves with constant acceleration and a circular trajectory with the target moving at constant speed. The constrained optimization includes besides the maneuver errors a metric of noise capacity strength, expressed by the mean square response of the filter to white noise. An optimal selection of α, β and γ parameters is provided for various penalties on the noise filtering.

1 Introduction

Target tracking and predicting is realized in *track-while-scan systems*, which are sampled data filters, based on previously observed positions containing measurement noise. The performance of these filters is a function of its noise smoothing behavior and its transient system response. However, these are competing metrics and a tradeoff is desired based on the objective of the designer.

A filter developed in the mid 50's, the α-β tracker, is popular because of its simplicity and therefore computational inexpensive requirements. This permits its use in limited power capacity applications like passive sonobuoys. The α-β filter performance has been described by Sklansky [1] and various improvements have followed ([2], [3] and [4]). Sklansky's early work delineated an optimization process with the dual objectives of minimizing the noise ratio and steady state maneuver error for a target on a circular trajectory.

Throughout this work closed form solutions are determined for the transient maneuver error, the steady state error and the noise ratio of an α-β-γ filter to gauge its tracking performance. These metrics are exploited to optimally select the set of the smoothing parameters which minimize the noise transmission capability and the tracking error. The evaluation of the maneuver error is based on certain target trajectories like straight line maneuvers and targets moving on a circular path which are used to construct various cost functions for the constrained optimization algorithm.

2 Performance Metrics

Unlike the α-β tracker, the α-β-γ filter is capable of tracking an accelerating target without steady-state error. Besides predicting the position, the velocity is also predicted by the α-β-γ filter. Its equations are given by:

$$x_p(k+1) = x_s(k) + Tv_s(k) + \frac{1}{2}T^2 a_s(k) \quad (1)$$

$$v_p(k+1) = v_s(k) + Ta_s(k) \ , \quad (2)$$

where the smoothed parameters are derived with the previous prediction and the weighted innovation as follows:

$$x_s(k) = x_p(k) + \alpha(x_o(k) - x_p(k)) \quad (3)$$

$$v_s(k) = v_p(k) + \frac{\beta}{T}(x_o(k) - x_p(k)) \quad (4)$$

$$a_s(k) = a_s(k-1) + \frac{\gamma}{2T^2}(x_o(k) - x_p(k)) \quad (5)$$

Applying the z-Transform to Equations (1) to (5) and solving for the ratio $\frac{x_p}{x_o}$ leads to the transfer function in z-domain which is

$$G(z) = \frac{x_p}{x_o} = \quad (6)$$

$$\frac{\alpha + (-2\alpha - \beta + \frac{1}{4}\gamma)z + (\alpha + \beta + \frac{1}{4}\gamma)z^2}{z^3 + (\alpha + \beta + \frac{1}{4}\gamma - 3)z^2 + (-2\alpha - \beta + \frac{1}{4}\gamma + 3)z + \alpha - 1}$$

The roots of the characteristic polynomial (CP), the denominator of the transfer function, are required to lie within the unit circle to guarantee stability. *Jury's Stability Test* [5] yields the constraints on the α, β and γ parameters as follows:

$$0 < \alpha < 2\ , \quad 0 < \beta\ , \quad 0 < \gamma \tag{7}$$

$$\beta < 4 - 2\alpha\ , \quad \gamma < \frac{4\alpha\beta}{2-\alpha} \tag{8}$$

This volume in the α-β-γ space defines the constraints for the subsequent optimization [6].

2.1 Noise Smoothing Performance

The noise filtering capability of the tracker is characterized by the noise ratio, defined as the ratio of the root mean square value (RMS) of the system response to the RMS value of the noisy input, which is:

$$\rho \equiv \sqrt{\frac{\overline{x_p^2}}{\overline{x_o^2}}} \tag{9}$$

Since, we require the tracker to reject measurement noise, a small value of ρ implies an excellent filtering of noise. If the input noise is assumed to be white noise, the following relationship can be derived [6]:

$$\begin{aligned}
\rho^2 &= \frac{-(4ba^2+2ab^2+4b^2)-4b(1+c)k_1}{2a(b+2a-4)b+a(1+c)k_2} \qquad (10)\\
k_1 &= k_{11}c^2+k_{12}c+k_{13}\\
k_{11} &= 4a+2a^3-ab+ba^2-6a^2\\
k_{12} &= ba^2+ab^2-6ab-2a^3+8a\\
k_{13} &= 6a^2-2ab^2-4ba^2+4a-2b^2+7ab\\
k_2 &= (b+2a-4)(c^2a-ca-c^2-2b+bc+1)\ ,
\end{aligned}$$

where the transformation into the a-b-c space has been introduced. In this space, the characteristic polynomial is represented as

$$(z+c)(z^2+(a+b-2)z+1-a), \tag{11}$$

where the second order factor has a form which is identical to the characteristic equation of the $\alpha-\beta$ filter and the third pole is real and is located at $-c$. Comparing the denominator of Equation (6) with Equation (11), the following transformation is derived:

$$\begin{aligned}
\alpha &= 1+c(1-a)\\
\beta &= a(1+c)+\frac{1}{2}b(1-c) \qquad (12)\\
\gamma &= 2b(1+c).
\end{aligned}$$

From Equation (12), we can infer that c equals -1 when $\gamma = 0$, and furthermore a and b degenerate to α and β.

Therefore, Equation (10) reduces for the α-β filter to the noise-ratio:

$$\rho^2 = \frac{2\alpha^2+\alpha\beta+2\beta}{\alpha(4-\beta-2\alpha)} \tag{13}$$

which differs from those derived by Sklansky [1] and Benedict and Bordner [2]. Equation (10) gauges the influence of noise on the output of the α-β-γ filter by using the unique transformation of Equation (12).

2.2 Tracking Maneuver Errors

Besides rejecting the noise, the maneuver error needs to be minimized. In this work, we endevour to derive closed form expressions for the steady state errors and the sum of the square of the errors which captures the transient behavior of the α-β-γ filter. This requires the specification of predefined target trajectories. The two classes of trajectories used in this work are straight line trajectories where the target is moving with constant acceleration and circular trajectories with the target moving at constant speed. The maneuver error is defined as the difference between the prediction and the observed position, which is called the innovation in the smoothing equations (3)-(5). Similar to the transfer function for the input-output relationship, the maneuver error transfer function can be derived as follows:

$$e_{\alpha\beta\gamma} = \frac{(z-1)^3 x_o}{\mathrm{CP}}\ . \tag{14}$$

Equation (14) reduces to the α-β filter by setting γ to zero.

$$e_{\alpha\beta} = \frac{(z-1)^2}{z^2+(\alpha+\beta-2)z+(1-\alpha)}x_o \tag{15}$$

2.2.1 Circular Trajectories:

Assuming that the target traces a circular path while moving at constant speed resulting in a constant angular velocity, the maneuver error can be resolved into components e_x and e_y in the x and y directions respectively. Substituting the circular path trajectory into Equation (14), we can show that the magnitude of the steady-state maneuver error is constant at all sampling instants, and only the phase is changing with each interval [1]. The magnitude of the steady-state error of a target on a circular path is given in Equation (16). To determine the steady state error for an α-β filter, we can substitute x_o in Equation (15), or set γ to zero in Equation (16). Both lead to the maneuver error for the α-β filter,

$$e = \frac{4R\sin^3(\frac{\omega T}{2})}{\sqrt{\sin^2(\frac{\omega T}{2})[((2-\alpha)\cos\omega T+\alpha+\beta-2)^2+\alpha^2\sin^2\omega T]+\gamma(\frac{\gamma}{32}(\cos\omega T+1)-\frac{\alpha}{4}\sin^2\omega T)}} \tag{16}$$

$$e = \frac{4R\sin^2(\frac{\omega T}{2})}{\sqrt{[(2-\alpha)\cos\omega T + \alpha + \beta - 2]^2 + \alpha^2 \sin^2 \omega T}}, \quad (17)$$

which is the same as that derived by Sklansky [1].

2.2.2 Straight Line Maneuver: Assuming the target trajectory to be a straight line, the coordinate system can always be placed such that one of the axes is coincident with the trajectory and the problem thus reduces to single dimension. The observed position for an accelerating target (a_s) with initial velocity (v) can now be determined in the z-domain, so that the definition of the error becomes:

$$e_{\alpha\beta\gamma} = \frac{\frac{1}{2}a_s T^2 z(z+1) + vTz(z-1)}{\mathrm{CP}} \; . \quad (18)$$

From Equation (18) the steady-state error can be derived by using the final value theorem [5]. Due to the acceleration, the α-β filter exhibits a steady-state error of:

$$e_{\alpha\beta}(t \to \infty) = \frac{a_s T^2}{\beta} \; , \quad (19)$$

whereas the steady-state error of the α-β-γ filter vanishes.

The transient response can be characterized by a metric which is defined as the sum of the square of the tracking errors as time tends to infinity. This metric can be defined in the discrete time domain as

$$J = \sum_{k=0}^{\infty} Te^2(k) \; . \quad (20)$$

It can be calculated by simulating the response of the transfer function given in Equation (18) and summing the resulting error. Instead, of using simulations, a closed form expression can be derived using *Parseval's Theorem* [5]. Similiar to the derivations of the noise-ratio, the α-β-γ filter is transformed to the a-b-c space, where the metric of Equation (20) can be shown to be:

$$\begin{aligned} J_{abc} \;\; = \;\; & \frac{2(c - ca - 1)T^3 v^2}{a(2a+b-4)(c-1)[c(a+b+ca)-(1-c)^2]} \\ & + \frac{\frac{1}{2}a_s^2 T^5 (c(a-1)-1)}{ab(1+c)[c(a+b+ca)-(1-c)^2]} . \quad (21) \end{aligned}$$

Equation (21) cannot be reduced to an α-β tracker by simply substituting c with -1 because this tracker exhibits a steady-state error for a target moving with constant acceleration. Consequently, solving for Equation (20) requires the subtraction of the steady-state value, defined in Equation (19). The transient response metric now reduces to:

$$\begin{aligned} J_{\alpha\beta} \;\; = \;\; & \frac{\frac{1}{4}a_s^2 T^5 (2\beta + \alpha\beta + 2\alpha^2)}{\alpha\beta^3} \\ & + \frac{v^2 T^3 (\alpha - 2)}{\alpha\beta(2\alpha+\beta-4)} + \frac{v a_s T^4}{\beta^2} \; . \quad (22) \end{aligned}$$

3 Design of Optimal Filters

Selection of the smoothing parameters within the region of stability is a function of the targets trajectory, the noise in the measurement, the steady state error, and the transient response of the filter. To arrive at the optimal set of parameters, a constrained parameter optimization problem is formulated. This is feasible since closed form expressions for various metrics have been derived in Section 2. Based on these metrics, closed form solutions for the optimal parameter are derived for certain target trajectories.

Defining a cost function which consists of two terms: the first, is a function of the tracker error and the second is a function of the noise ratio. This provides us with the flexibility to include steady state error, transient error etc. and permits us to weight them based on their importance. Thus, the cost function becomes:

$$f \equiv f(e, \rho, \kappa) \quad (23)$$

where κ is a weighting factor to penalize the contribution of one term relative to the other. The constraints for the optimization are defined by the stability region of the tracker defined by Equations (7) and (8). Different metrics have been derived to capture the tracking error for specific maneuvers such as straight line (Equation (21)) or circular target path (Equation (16)). The choice of the appropriate cost function depends on the goal of the optimization.

3.1 Circular Trajectory

To study the variation of the smoothing parameters as a function of the relative importance of the steady state error for a circular maneuver and the noise ratio, a series of optimizations have been carried out. The maneuver error e in Equation (23) is obtained for a circular target path in steady state conditions. Substituting Equation (17) and (13) in the cost function f, Equation (23) leads to the cost function

$$J = e^2 + \kappa\rho^2\overline{x_o^2}. \quad (24)$$

Assuming that the smallest turn radius of a submarine, for instance, with $v = 3$ knots speed is $45\,m$, leads to the angular velocity of $\omega = v/R = 0.034$ rad/sec, and furthermore, assume that the variance of the measurement noise is 200 m^2. To illustrate the effect of changing the weighting parameter, a series of optimizations are carried out and the resulting set of α, β and γ respectively parameters are plotted. Figure (1) illustrates that the optimal set of parameters monotonically decrease with increasing κ

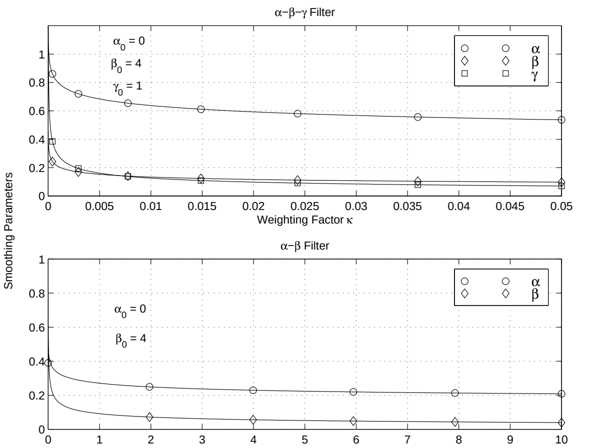

Figure 1: Optimal Solution of α, β and γ for tracking a circular path

until a final value for large penalty on the noise-ratio is reached. The shape of the curve does not change for circular maneuvers of different radii, since the increase in the radius of the trajectory is equivalent to increasing the weighting factor.

3.2 Straight Line Maneuver

In practice the tracker rarely reaches steady state because the target path is continuously changing. Including the transient response of the tracker in the cost function leads to a more realistic metric. A closed form solution is given for an α-β-γ and an α-β tracker in Equations (21) and (22). We can now modify Equation (24) by including the metric which measures the transient performance, as follows:

$$f = J_{\alpha\beta(\gamma)} + \kappa_{\mathrm{SS}} e^2_{\alpha\beta(\gamma)}(t \to \infty) + \kappa\rho^2\overline{x_o^2} \; , \qquad (25)$$

where $J_{\alpha\beta(\gamma)}$ is the transient error metric and $e^2_{\alpha\beta(\gamma)}(t \to \infty)$ the steady state error of the α-β tracker or the α-β-γ tracker respectively. The weighting factor κ_{SS} adjusts the influence for the steady state error. The effect of the velocity and acceleration on the optimal set of the smoothing parameter of an α-β tracker is shown in Figure (2). The three curves in the graph correspond to objecitve functions which weight the noise ratio to the tracking errors for targets moving with constant velocity, constant acceleration and steady state tracking error for a constant acceleration input.

The solid line in Figure (2) exhibits variation of the optimal parameters as a function of the weighting parameter κ for targets with constant velocity. A closed form expression of this optimal curve can be derived as follows. The objective function in Equation (25) reduces for the proposed case to:

$$f = J_{\alpha\beta,v} + \kappa\rho^2\overline{x_o^2} \; , \qquad (26)$$

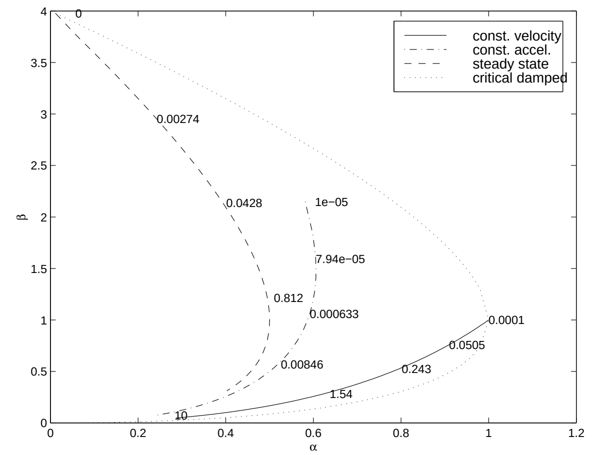

Figure 2: Optimal Solution of α and β for a straight line maneuver with certain values of κ

where $J_{\alpha\beta,v}$ is the transient cost for constant velocity. The optimal solution is obtained by searching for the parameters where the gradient of f vanishes, which can be rewritten as:

$$\begin{bmatrix} \frac{\partial J_{\alpha\beta,v}}{\partial\alpha} & \frac{\partial(\rho^2)}{\partial\alpha} \\ \frac{\partial J_{\alpha\beta,v}}{\partial\beta} & \frac{\partial(\rho^2)}{\partial\beta} \end{bmatrix} \begin{bmatrix} 1 \\ \kappa\overline{x_o^2} \end{bmatrix} = 0 \; . \qquad (27)$$

Equation (27) can be satisfied only if the determinant of the Jacobian matrix is zero since the vector $[1 \quad \kappa\overline{x_o^2}]'$ never vanishes. Equating the determinant to zero, we can solve for β resulting in the equation:

$$\beta = \frac{\alpha^2}{2-\alpha} \; , \qquad (28)$$

which matches the optimal solution proposed by Benedict and Bordner [2]. In addition to Equation (28), the optimal set of the parameters α and β require satisfaction of one of Equations 27, which for instance, is

$$0 = [2\beta(4\beta\alpha + 4\alpha^2 - 4\beta + \beta^2)]\kappa + 2v^2T^3(-\alpha^2 + 4\alpha - 4 + \beta) \; . \qquad (29)$$

This equation determines the location of the optimal parameters on the curve given by Equation (28) as a function of the weighting factor κ. In Figure (2) some values of the weighting factor are shown. Very small penalty on the noise ratio optimizes the settling time of the tracker, which is, of course, the shortest if the two poles lie at $z = 0$. This leads to an infinitisimaly quick tracker response and is given at the smoothing parameters $\alpha = \beta = 1$. Increasing the penalty of the noise ratio, moves the parameter α and β towards the critically damped curve.

The optimal curve for constant acceleration is obtained by setting the initial velocity and the steady state weight to zero. Optimizing with respect to the transient response of accelerating targets, results in higher values for β as shown by the dash-dot line in Figure (2). The equation describing this curve is obtained by applying the same algorithm as for the case of constant velocity maneuver. It can be shown that the dash-dot line in Figure (2) is described by:

$$\beta_{1,2} = 3 - \frac{5}{2}\alpha \pm \frac{1}{2}\sqrt{(36 - 60\alpha + \alpha^2)} \tag{30}$$

and the second equation that needs to be satisfied is:

$$0 = [4\beta^3(4\alpha^2 - 4\beta + \beta^2 + 4\beta\alpha)]\kappa + a_s^2(\alpha^2 - \beta)(2\alpha - 4 + \beta)^2 \,. \tag{31}$$

The dashed line in Figure (2), which corresponds to a cost function which includes the steady state error, reveals that minimizing the steady state error of the tracker by increasing the weight κ_{SS}, forces β to maximize. The two equations describing the optimal solutions for the case where the objective function has a strong penalty on the steady state error are derived as follows:

$$\begin{aligned} \beta_{1,2} &= 2 - 2\alpha \pm 2\sqrt{(1 - 2\alpha)} && (32) \\ 0 &= [(4\alpha^2 - 4\beta + \beta^2 + 4\beta\alpha)]2\kappa && (33) \end{aligned}$$

Figure (2) reveals that a better noise smoothening is obtained with smaller values of β and α, conversely, faster response is obtained with higher values of β. The special case of constant velocity has its fastest response at $\alpha = \beta = 1$. Varying the weighting factor κ can also be interpreted in changing the sensor noise variance since both parameters multiply the noise-ratio.

Similiar to the α-β filter, optimal sets of the smoothing parameter of an α-β-γ filter are derived and may be display likewise in the upper plot of Figure 1.

A closed form expression of the optimal set of parameters can be derived for a straight line target trajectory where the target is accelerating. Constructing a matrix of the gradients of the terms of the cost function with respect to a, b and c and equating the determinant to zero and solving for the parameter b, leads to the optimal solution in the a-b-c space:

$$\begin{aligned} b_1 &= \frac{(1+c)^2(a-1) + a^2 c}{c(2-a)} && (34) \\ b_2 &= \frac{(1+c)(c+1-a)}{c} && (35) \end{aligned}$$

It can be seen that b_1 reduces to the optimal solution of the α-β filter (Equation (28)) if $c = -1$, whereas b_2 vanishes for this case. The two additional conditions to be satisfied to arrive at the optimal values of the parameters cannot easily be simplified.

4 Conclusion

This paper focuses on the optimal design of α-β-γ filters. To quantify the performance of these filters, various metrics are defined such as noise-ratio, steady-state maneuver error and transient response metrics. Closed form solutions are derived in the a-b-c space which can be transformed into the α-β-γ space via a nonlinear transformation. The maneuver errors are defined for specific target trajectories like straight line and circular path maneuvers. These are subsequently used in conjunction with the noise-ratio to determine a figure of demerit. In particular, a constrained parameter optimization problem is formulated, where α, β and γ are bounded to lie within the stability volume. Variation of the tracker parameters for different weights of the cost function are studied to provide the designer with information for the optimal selection of α, β and γ parameters. Furthermore, optimal closed form solutions are derived for steady-state circular and straight line path maneuvers, where straight line maneuvers have the target moving at constant acceleration or constant velocity.

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References

[1] Jack Sklansky. Optimizing the dynamic parameter of a track-while-scan system. *RCA Laboratories, Princton, N.J.*, June 1957.

[2] T. R. Benedict and G. W. Bordner. Synthesis of an optimal set of radar track-while-scan smoothing equations. In *IRE Transaction on Automatic Control*, volume AC-1, July 1962.

[3] Paul R. Kalata. $\alpha - \beta$ target tracking systems: A survey. In *American Control Conference/WM12*. ECE Department, Drexel Univeristy Philadelphia, Pennsylvania, 1992.

[4] W. D. Blair. Two-stage alpha-beta-gamma estimator for tracking maneuvering targets. In *American Control Conference/WM12*. Weapons Control Division, Naval Surface Warfare Center, Dahlgren, Virginia, 1992.

[5] Katsuhiko Ogata. *Discrete-Time Control Systems*. Prentice-Hall,Engelwood Cliffs, New Jersey, University of Minnesota, 1987.

[6] Dirk Tenne. Synthesis of target-track estimators. Master's thesis, Department of Mechanical & Aerospace Engineering, State University of New York at Buffalo, 1998.